

ISOMETRIC PROPERTIES OF RELATIVE BOUNDED COHOMOLOGY

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ABSTRACT. We show that the morphism induced by the inclusion of pairs $(X, \emptyset) \subset (X, Y)$ between the relative bounded cohomology of (X, Y) and the bounded cohomology of X is an *isometric* isomorphism in degree at least 2 if the fundamental group of each connected component of Y is amenable. As an application, we provide a self-contained proof of Gromov's Equivalence Theorem and a generalization of a result by Fujiwara and Manning on the simplicial volume of generalized Dehn fillings.

1. INTRODUCTION

In the mid seventies, Gromov introduced the bounded cohomology of a space and showed that it vanishes in all degrees $n \geq 1$ for simply connected CW-complexes [5]. Brooks pointed out that this implies that the bounded cohomology of a space is isomorphic to the one of its fundamental group [1]. In the latter note the author also made the first step towards the relative homological algebra approach to the bounded cohomology of groups. Ivanov then developed this approach (with trivial coefficients) [6], incorporating the seminorm into the theory. This led to the final form of Gromov's theorem, namely that for a countable CW-complex the bounded cohomology is isometrically isomorphic to the bounded cohomology of its fundamental group. We emphasize that, here and in the sequel, the coefficients are the trivial module \mathbb{R} .

After Gromov's seminal paper [5], bounded cohomology has admitted many generalizations and applications in a variety of contexts, though, ever since Ivanov's proof of Gromov's theorem, a sore point in the theory has been to establish that a given isomorphism is isometric. In this note, we will study this question for relative homology and bounded cohomology.

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Bounded cohomology can be defined for pairs (X, Y) of spaces, where Y is a subspace of the space X , and there is an exact sequence

$$\dots \longrightarrow H_b^{n-1}(Y) \xrightarrow{\delta^n} H_b^n(X, Y) \xrightarrow{j^n} H_b^n(X) \xrightarrow{i^n} H_b^n(Y) \longrightarrow \dots,$$

where j^n is induced by the inclusion of the corresponding cochain complexes, i^n is induced by the restriction map and δ^n is the connecting homomorphism.

A striking consequence of this long exact sequence can be obtained when we assume that each connected component of Y has amenable fundamental group. Indeed, as observed by Trauber in the 70's, one of the characteristic features of bounded group cohomology is that it vanishes for amenable groups in degree $n \geq 1$. This implies that j^n is an isomorphism of vector spaces for $n \geq 2$. In low degree, the isomorphism does not hold. Instead, it follows from $H_b^1(X) = 0$ that we have an exact sequence

$$H_b^0(X, Y) \xhookrightarrow{j^0} H_b^0(X) \xrightarrow{i^0} H_b^0(Y) \xrightarrow{\delta^1} H_b^1(X, Y).$$

If X is path connected then $H_b^0(X, Y) = 0$ and $H_b^0(X) = \mathbb{R}$, while $H_b^0(Y) = \ell^\infty(\pi_0(Y))$. Here and in the sequel, $\ell^\infty(S)$ is the Banach space of all bounded real valued functions on the set S .

Our main result is that, under the above hypotheses, j^n is isometric:

THEOREM 1. *Let $X \supseteq Y$ be a pair of countable CW-complexes. Assume that each connected component of Y has amenable fundamental group. Then the morphism obtained from the inclusion*

$$j^n : H_b^n(X, Y) \longrightarrow H_b^n(X)$$

is an isometric isomorphism for every $n \geq 2$.

Gromov's Equivalence Theorem. Beside the obvious ℓ^1 -seminorm, the relative homology of a pair of spaces can be endowed with a whole one parameter family of seminorms introduced by Gromov [5, Section 4.1]. Indeed, let (X, Y) be a pair of topological spaces and take a singular chain $c \in C_n(X)$ for $n \in \mathbb{N}$. Then, for every $\theta \geq 0$, one can define a norm on $C_n(X)$ by setting

$$\|c\|_1(\theta) = \|c\|_1 + \theta \|d_n c\|_1.$$

Taking the infimum value over the suitable sets of representatives, this norm induces a seminorm on the relative homology module $H_\bullet(X, Y)$, which is still denoted by $\|\cdot\|_1(\theta)$. Notice that, for every $\theta \in [0, \infty)$, the norm $\|\cdot\|_1(\theta)$ is equivalent, but not equal, to the usual ℓ^1 -norm $\|\cdot\|_1 = \|\cdot\|_1(0)$ on $H_\bullet(X, Y)$. By passing to the limit, one can also define the seminorm $\|\cdot\|_1(\infty)$, which however need not be equivalent to $\|\cdot\|_1$. For example, $\|\alpha\|_1(\infty) = \infty$ when $\alpha \in H_n(X, Y)$ is such that $\|\partial_n \alpha\|_1 > 0$, where $\partial_n : H_n(X, Y) \rightarrow H_{n-1}(Y)$ is the connecting homomorphism of the sequence of the pair.

The following result is stated by Gromov in [5] (see also Remark 4.2 for a comment about Gromov's original statement). However, Gromov's proof of Theorem 2 is not carried out in details and relies on the rather technical theory of multicomplexes. In Section 4 we provide a complete and direct proof of Theorem 2 as a consequence of Theorem 1.

THEOREM 2 (Equivalence Theorem, [5, page 57]). *Let $X \supseteq Y$ be a pair of countable CW-complexes. If the fundamental groups of all connected components of Y are amenable, then the seminorms $\|\cdot\|_1(\theta)$ on $H_n(X, Y)$, for $n \geq 2$, are equal for every $\theta \in [0, \infty]$.*

In [10], Park uses a mapping cone construction to compute the relative ℓ^1 -homology of topological pairs, and endows this ℓ^1 -homology with a one parameter family of seminorms. This approach may also be exploited in the case of singular homology, and in this context it is not difficult to show that Park's seminorms coincide with Gromov's. A dual mapping cone construction is then used in [11] to define a one parameter family of dual seminorms on relative bounded cohomology¹. The arguments developed in Section 4 for the proof of Theorem 2, which are inspired by Park's approach, may further be refined to prove that Park's seminorms on cohomology coincide with the usual Gromov seminorm, provided that the map $H_b^\bullet(X, Y) \rightarrow H_b^\bullet(X)$ is an isometric isomorphism². Together with Theorem 1 and a standard duality argument, this fact can be exploited to provide another proof of Theorem 2.

As noticed by Gromov, Theorem 2 admits the following equivalent formulation, which is inspired by Thurston [12, Section 6.5]:

THEOREM 3. *Let $X \supseteq Y$ be a pair of countable CW-complexes, and suppose that the fundamental groups of all the components of Y are amenable. Let $\alpha \in H_n(X, Y)$, $n \geq 2$. Then, for every $\epsilon > 0$ there exists a representative $c \in C_n(X)$ of α such that $\|c\|_1 < \|\alpha\|_1 + \epsilon$ and $\|d_n c\|_1 < \epsilon$.*

Theorem 3 plays an important role in several results about the (relative) simplicial volumes of glueings and fillings. In Section 5 we provide a proof of the equivalence between the statements of Theorem 2 and 3.

Let (X, Y) satisfy the hypothesis of Theorem 3, and let $n \geq 2$. Via Löh's "translation mechanism" [7], Theorem 1 implies that the natural map $H_n^{\ell^1}(X) \rightarrow H_n^{\ell^1}(X, Y)$ on ℓ^1 -homology is an isometric isomorphism. As a consequence, since the maps $H_n(X) \rightarrow H_n^{\ell^1}(X)$ and $H_n(X, Y) \rightarrow H_n^{\ell^1}(X, Y)$ induced by the inclusions of singular chains in ℓ^1 -chains are norm preserving [7], the homology map

¹Unless otherwise stated, we understand that relative bounded cohomology is endowed with the seminorm introduced by Gromov in [5, 9, Section 4.1], which is induced by the ℓ^∞ -norm on relative cochains (see also Section 2). This is the case, for example, in the statement of Theorem 1.

²In general, Park's seminorms are different from Gromov's [2, Proposition 6.4].

$j_n: H_n(X) \rightarrow H_n(X, Y)$ is norm preserving, although it is surely not an isomorphism in general. This implies that every class lying in $j_n(H_n(X)) \subseteq H_n(X, Y)$ satisfies the conclusion of Theorem 3.

In the general case, the isometric isomorphism between $H_n^{\ell^1}(X, Y)$ and $H_n^{\ell^1}(X)$ ensures that every ordinary relative homology class $\alpha \in H_n(X, Y)$ may be represented (in the corresponding relative ℓ^1 -homology module) by an absolute ℓ^1 -cycle c whose norm is close to $\|\alpha\|_1$. One may wonder whether the finite approximations c_i of c may be used to construct the representative required in Theorem 3, since the ℓ^1 -norm of $d_n c_i$ is approaching zero as i tends to infinity. However, it is not clear how to control the support of $d_n c_i$, which may not be contained in Y .

Simplicial volume of generalized Dehn fillings. Let M be the natural compactification of a complete finite volume hyperbolic n -manifold with toric cusps. A *generalized Dehn filling* of M was defined by Fujiwara and Manning in [3] as the space obtained by replacing the cusps of the interior of M with compact partial cones of their boundaries (see Section 6 for a precise definition). Moreover, in [4] they proved that the simplicial volume does not increase under generalized Dehn filling. Note that in dimension 3, the notion of generalized Dehn filling coincides with the usual notion of Dehn filling, and the fact that the (relative) simplicial volume of any cusped hyperbolic 3-manifold strictly decreases under Dehn filling is a classical result by Thurston [12].

Fujiwara and Manning's argument easily extends to the case in which the fundamental group of M is residually finite and the inclusion of each boundary torus in M induces an injective map on fundamental groups. Recall that these conditions are always fulfilled if the interior of M is a complete finite-volume hyperbolic manifold. In Section 6 we generalize Fujiwara and Manning's result to the case of an arbitrary manifold with toric boundary:

THEOREM 4. *Let M be a compact orientable n -manifold with boundary given by a union of tori, and let N be a generalized Dehn filling of M . Then*

$$\|N\| \leq \|M, \partial M\|.$$

We provide two slightly different proofs of Theorem 4. The first one makes use of the Equivalence Theorem (or more precisely its reformulation given in Theorem 3), the other one relies directly on Theorem 1.

2. RESOLUTIONS IN BOUNDED COHOMOLOGY

Let X be a space, where here and in the sequel by a space we will always mean a countable CW-complex. We denote by $C_b^n(X)$ the complex of bounded real valued n -cochains on X and, if $Y \subset X$ is a subspace, by $C_b^n(X, Y)$ the subcomplex of those bounded cochains that vanish on simplices with image contained in Y . All these spaces of cochains are endowed with the ℓ^∞ -norm and the corresponding cohomology groups are equipped with the corresponding quotient seminorm.

For our purposes, it is important to observe that the universal covering map $p : \tilde{X} \rightarrow X$ induces an isometric identification of the complex $C_b^n(X)$ with the complex $C_b^n(\tilde{X})^\Gamma$ of $\Gamma := \pi_1(X)$ -invariant bounded cochains on \tilde{X} . Similarly, if $Y' := p^{-1}(Y)$, we obtain an isometric identification of the complex $C_b^n(X, Y)$ with the complex $C_b^n(\tilde{X}, Y')^\Gamma$ of Γ -invariants of $C_b^n(\tilde{X}, Y')$.

The main ingredient in the proof of Theorem 1, which is also essential in the proof of Gromov's theorem, is the result of Ivanov [6] that the complex of Γ -invariants of

$$\mathbb{R} \longrightarrow C_b^0(\tilde{X}) \longrightarrow C_b^1(\tilde{X}) \longrightarrow \dots$$

computes the bounded cohomology of Γ . In fact, we will use the more precise statement that the latter complex is a strong resolution of \mathbb{R} by relatively injective Γ -Banach modules (see [6] for the definitions of strong resolutions and relatively injective modules).

By standard homological algebra techniques [6], it follows from the fact that $C_b^n(\tilde{X})$ is a strong resolution by Γ -modules and $\ell^\infty(\Gamma^{\bullet+1})$ is a cochain complex (even a strong resolution) by relatively injective Γ -modules that there exists a Γ -morphism of complexes

$$(\diamond) \quad g^n : C_b^n(\tilde{X}) \longrightarrow \ell^\infty(\Gamma^{n+1})$$

extending the identity, and such that g^n is contracting, i.e. $\|g^n\| \leq 1$, for $n \geq 0$. This map induces Ivanov's isometric isomorphism $H_b^\bullet(X) \rightarrow H_b^\bullet(\Gamma)$.

The second result we need lies at the basis of the fact that the bounded cohomology of Γ can be computed isometrically from the complex of bounded functions on any amenable Γ -space. We will need only a particular case of the isomorphism, which is the existence of a contracting map between the complex $\ell^\infty(\Gamma^{n+1})$ and the complex of alternating bounded functions $\ell_{\text{alt}}^\infty(S^{n+1})$ when S is a discrete amenable Γ -space. This is a very special case of [9], for which we present a direct proof.

PROPOSITION 2.1 ([9, Theorem 7.2.1]). *Assume that Γ is a group acting on a set S such that all stabilizers are amenable subgroups of Γ . Then for $n \geq 0$ there is a Γ -morphism of complexes*

$$\mu^n : \ell^\infty(\Gamma^{n+1}) \longrightarrow \ell_{\text{alt}}^\infty(S^{n+1})$$

extending $\text{Id}_\mathbb{R} : \mathbb{R} \rightarrow \mathbb{R}$ that is contracting.

Proof. Alternation gives a contracting Γ -morphism of complexes

$$\ell^\infty(S^{n+1}) \longrightarrow \ell_{\text{alt}}^\infty(S^{n+1}),$$

so that it suffices to construct a contracting Γ -morphism of complexes

$$\mu^n : \ell^\infty(\Gamma^{n+1}) \longrightarrow \ell^\infty(S^{n+1}).$$

We first construct μ^0 and then inductively μ^n , for $n \geq 1$. Identify S with a disjoint union $\sqcup_{i \in I} \Gamma/\Gamma_i$ of right cosets, where $\Gamma_i < \Gamma$ is amenable and let $\lambda_i \in \ell^\infty(\Gamma_i)^*$ be a left Γ_i -invariant mean, for every $i \in I$. We define $\mu^0 : \ell^\infty(\Gamma) \rightarrow \ell^\infty(S)$ for $f \in \ell^\infty(\Gamma)$ by setting $\mu^0(f)(\gamma\Gamma_i)$ to be the Γ_i -invariant mean λ_i of the bounded function $\Gamma_i \rightarrow \mathbb{R}$ defined by $\eta \mapsto f(\gamma\eta)$. Clearly $\mu^0(\mathbf{1}_\Gamma) = \mathbf{1}_S$, so that μ^0 extends $\text{Id}_\mathbb{R} : \mathbb{R} \rightarrow \mathbb{R}$ and $\|\mu^0\| \leq 1$.

Assume now that we have defined $\mu^{n-1} : \ell^\infty(\Gamma^n) \rightarrow \ell^\infty(S^n)$. Then we define μ^n as the composition of the following maps:

$$\begin{array}{ccccc}
 \ell^\infty(\Gamma^{n+1}) & \xrightarrow{=} & \ell^\infty(\Gamma \times \Gamma^n) & \xrightarrow{\cong} & \ell^\infty(\Gamma, \ell^\infty(\Gamma^n)) \\
 \vdots & & & & \downarrow \\
 & & & & \ell^\infty(\Gamma, \ell^\infty(S^n)) \\
 \vdots & & & & \downarrow \cong \\
 \mu^n \downarrow & & & & \ell^\infty(S^n, \ell^\infty(\Gamma)) \\
 \vdots & & & & \downarrow \\
 \ell^\infty(S^{n+1}) & \xleftarrow{=} & \ell^\infty(S \times S^n) & \xleftarrow{} & \ell^\infty(S^n, \ell^\infty(S)),
 \end{array}$$

where \cong denotes a Banach space isomorphism, while the first vertical arrow is induced by $\mu^{n-1} : \ell^\infty(\Gamma^n) \rightarrow \ell^\infty(S^n)$ and the third by $\mu^0 : \ell^\infty(\Gamma) \rightarrow \ell^\infty(S)$. Since all morphisms involved are contracting and equivariant for suitable Γ -actions, the same holds for μ^n . Finally one verifies that $(\mu^n)_{n \geq 0}$ is a morphism of complexes. \square

3. PROOF OF THEOREM 1

Let, as above, $p : \tilde{X} \rightarrow X$ be the universal covering map, $\Gamma := \pi_1(X)$ and $Y = \sqcup_{i \in I} C_i$ the decomposition of Y into a union of connected components. If \check{C}_i is a choice of a connected component of $p^{-1}(C_i)$ and Γ_i denotes the stabilizer of \check{C}_i in Γ then

$$p^{-1}(C_i) = \bigsqcup_{\gamma \in \Gamma/\Gamma_i} \gamma \check{C}_i.$$

Let $\mathcal{F} \subset \tilde{X} \setminus Y'$ be a fundamental domain for the Γ -action on $\tilde{X} \setminus Y'$, where $Y' = p^{-1}(Y)$ as before. Define the Γ -equivariant map

$$r : \tilde{X} \rightarrow S := \Gamma \sqcup \bigsqcup_{i \in I} \Gamma/\Gamma_i$$

as follows:

$$r(\gamma x) := \begin{cases} \gamma \in \Gamma & \text{if } x \in \mathcal{F}, \\ \gamma \Gamma_i \in \Gamma/\Gamma_i & \text{if } x \in \check{C}_i. \end{cases}$$

For every $n \geq 0$ define

$$r^n : \ell_{\text{alt}}^\infty(S^{n+1}) \longrightarrow C_{\text{b}}^n(\tilde{X})$$

by

$$r^n(c)(\sigma) = c(r(\sigma_0), \dots, r(\sigma_n)),$$

where $c \in \ell_{\text{alt}}^\infty(S^{n+1})$ and $\sigma_0, \dots, \sigma_n \in \tilde{X}$ are the vertices of a singular simplex $\sigma : \Delta^n \rightarrow \tilde{X}$. Clearly $(r^n)_{n \geq 0}$ is a Γ -morphism of complexes extending the identity on \mathbb{R} and $\|r^n\| \leq 1$ for all $n \geq 0$.

Observe that if $n \geq 1$ and $\sigma(\Delta^n) \subset Y'$, then there are $i \in I$ and $\gamma \in \Gamma$ such that $\sigma(\Delta^n) \subset \gamma\check{C}_i$. Thus

$$r(\sigma_0) = \dots = r(\sigma_n) = \gamma\Gamma_i$$

and thus

$$r^n(c)(\sigma) = c(\gamma\Gamma_i, \dots, \gamma\Gamma_i) = 0,$$

since c is alternating. This implies that the image of r^n is in $C_{\text{b}}^n(\tilde{X}, Y')$. Thus we can write $r^n = j^n \circ r_1^n$, where $j^n : C_{\text{b}}^n(\tilde{X}, Y') \hookrightarrow C_{\text{b}}^n(\tilde{X})$ is the inclusion and $r_1^n : \ell_{\text{alt}}^\infty(S^{n+1}) \rightarrow C_{\text{b}}^n(\tilde{X}, Y')$ is a norm decreasing Γ -morphism that induces a norm non-increasing map³ in cohomology

$$H(r_1^n) : H^n(\ell_{\text{alt}}^\infty(S^{\bullet+1})^\Gamma) \longrightarrow H_{\text{b}}^n(X, Y),$$

for $n \geq 1$.

Using the map g^n defined in (\diamond) and the map μ^n provided by Proposition 2.1 since, for all i , the group Γ_i is a quotient of $\pi_1(C_i)$, and hence amenable, we have the following diagram

$$\begin{array}{ccccc} C_{\text{b}}^n(\tilde{X}) & \xrightarrow{g^n} & \ell^\infty(\Gamma^{n+1}) & \xrightarrow{\mu^n} & \ell_{\text{alt}}^\infty(S^{n+1}) & \xrightarrow[r_1^n]{\text{for } n \geq 1} & C_{\text{b}}^n(\tilde{X}, Y') \\ & \searrow & & \searrow & \searrow & \searrow & \downarrow j^n \\ & & & & & & C_{\text{b}}^n(\tilde{X}), \\ & & \text{extends } \text{Id}_{\mathbb{R}} & & & & \end{array}$$

where the dotted map is the composition $r^n \circ \mu^n \circ g^n$ which is a Γ -morphism of strong resolutions by relatively injective modules extending the identity, and hence induces the identity on $H_{\text{b}}^n(X) = H^n(C_{\text{b}}^\bullet(\tilde{X})^\Gamma)$.

We proceed now to show that, for $n \geq 2$, the map

$$H(j^n) : H_{\text{b}}^n(X, Y) \longrightarrow H_{\text{b}}^n(X)$$

induced by j^n is an isometric isomorphism in cohomology. In view of the long exact sequence for pairs in bounded cohomology and the fact that $H_{\text{b}}^\bullet(Y) = 0$ in degree greater than 1, we already know that $H(j^n)$ is an isomorphism. Let us

³To avoid confusion, henceforth we use a different notation for the chain and cochain maps and the induced homology and cohomology maps. This is contrary to our notation in the introduction.

denote by ψ^n the map induced in cohomology by the composition $r_1^n \circ \mu^n \circ g^n$. From the above it follows that

$$H(j^n) \circ \psi^n = \text{Id}_{H_b^n(X)} .$$

Let $y \in H_b^n(X, Y)$ and set $x = H(j^n)(y)$. Then $H(j^n)(\psi^n(x)) = x$ and, as $H(j^n)$ is injective, we get $y = \psi^n(x)$. Since the maps $H(j^n)$ and ψ^n are norm non-increasing it follows that

$$\|x\|_\infty = \|H(j^n)(y)\|_\infty \leq \|y\|_\infty \quad \text{and} \quad \|y\|_\infty = \|\psi^n(x)\|_\infty \leq \|x\|_\infty$$

so that $\|H(j^n)(y)\|_\infty = \|x\|_\infty = \|y\|_\infty$ and hence $H(j^n)$ is norm preserving.

4. PROOF OF THEOREM 2

Recall from the introduction that Gromov endowed the homology module $H_n(X, Y)$ with a one parameter family of seminorms $\|\cdot\|_1(\theta)$, $\theta \in [0, \infty]$. By definition, $\|\cdot\|_1(0)$ is equal to the usual ℓ^1 -seminorm $\|\cdot\|_1$, while $\|\cdot\|_1(\infty)$ is defined by taking the limit of $\|\cdot\|_1(\theta)$ as θ tends to infinity. Therefore, in order to prove Theorem 2 it is sufficient to show that, if every component of Y has amenable fundamental group, then $\|\cdot\|_1(\theta) = \|\cdot\|_1$ for every $\theta \in (0, \infty)$.

As is customary, in order to compare seminorms in homology we will compare cocycles in bounded cohomology, and exploit the duality between homology and cohomology that is usually provided, in this context, by the Hahn-Banach Theorem (see *e.g.* [7] for a detailed discussion of this issue).

In what follows, if $c \in C_n(X)$ is a representative of a class $\alpha \in H_n(X, Y)$, with a slight abuse of notation $d_n c$ will be used to identify both the element $d_n c \in C_{n-1}(X)$ and its preimage in $C_{n-1}(Y)$ via the inclusion $i_{n-1}: C_{n-1}(Y) \rightarrow C_{n-1}(X)$. As mentioned in the introduction, the proof of the following result is inspired by some techniques developed by Park in [10] and [11].

PROPOSITION 4.1. *Let $\theta \in (0, \infty)$ and take $\alpha \in H_n(X, Y)$. Then there exist $f \in C_b^n(X)$, $g \in C_b^{n-1}(Y)$ such that the following conditions hold:*

- (1) $d^n f = 0$ and $i^n(f) = -d^{n-1} g$;
- (2) $f(c) + g(d_n c) = \|\alpha\|_1(\theta)$ for every representative $c \in C_n(X)$ of α ;
- (3) $\|f\|_\infty \leq 1$.

Proof. Let us consider the direct sum

$$V = C_n(X) \oplus C_{n-1}(Y) ,$$

and endow V with the norm $\|\cdot\|_1(\theta)$ defined by

$$\|(u, v)\|_1(\theta) = \|u\|_1 + \theta \|v\|_1 .$$

Let us also set

$$V^* = C_b^n(X) \oplus C_b^{n-1}(Y) ,$$

and endow V^* with the ℓ^∞ -norm $\|\cdot\|_\infty(\theta)$ defined by

$$\|(f, g)\|_\infty(\theta) = \max\{\|f\|_\infty, \theta^{-1}\|g\|_\infty\} .$$

It is readily seen that $(V^*, \|\cdot\|_\infty(\theta))$ is isometrically identified to the topological dual of $(V, \|\cdot\|_1(\theta))$ via the pairing

$$V^* \times V \rightarrow \mathbb{R}, \quad ((f, g), (u, v)) \mapsto f(u) + g(v).$$

Let $B_n(X) \subseteq C_n(X)$ be the space of absolute n -boundaries of X , and let us set $W_1 = B_n(X) \oplus \{0\} \subseteq V$. We also set

$$W_2 = \{(u, v) \in V \mid u = i_n(z), v = d_n z \text{ for some } z \in C_n(Y)\} \subseteq V,$$

and $W = W_1 + W_2$. It is easy to verify that two relative cycles $c, c' \in C_n(X)$ represent the same element in $H_n(X, Y)$ if and only if $(c, d_n c) - (c', d_n c')$ lies in W . Let $c \in C_n(X)$ be any representative of $\alpha \in H_n(X, Y)$. Our previous remark implies that

$$(4.1) \quad \|\alpha\|_1(\theta) = \inf\{\|(c, d_n c) - w\|_1(\theta) \mid w \in W\} = \text{dist}((c, d_n c), W),$$

where the last distance is computed of course with respect to the norm $\|\cdot\|_1(\theta)$ on V .

Now, an easy application of the Hahn-Banach Theorem ensures that we may find a functional $(f, g) \in V^*$ such that the following conditions hold:

- (a) $0 = (f, g)(u, v) = f(u) + g(v)$ for every $(u, v) \in W$;
- (b) $(f, g)(c, d_n c) = f(c) + g(d_n c) = \text{dist}((c, d_n c), W) = \|\alpha\|_1(\theta)$;
- (c) $\|(f, g)\|_\infty(\theta) = 1$.

The fact that (f, g) vanishes on W_1 implies that $d^n f = 0$, while $(f, g)|_{W_2} = 0$ implies that $i^n(f) = -d^{n-1}g$, so (a) implies point (1) of the statement. Point (1) implies in turn that $f(c') + g(d_n c') = f(c) + g(d_n c)$ for every representative $c' \in C_n(X)$ of α , so point (2) is a consequence of (b). Since $\|f\|_\infty \leq \|(f, g)\|_\infty(\theta)$, point (3) is a consequence of (c). \square

We are now ready to prove Theorem 2. Suppose that the fundamental group of every component of Y is amenable, let $n \geq 2$ and take an element $\alpha \in H_n(X, Y)$. Also take $\theta \in (0, \infty)$. Since the inequality $\|\alpha\|_1 \leq \|\alpha\|_1(\theta)$ is obvious, we need to show that

$$\|\alpha\|_1(\theta) \leq \|\alpha\|_1.$$

Let $f \in C_b^n(X)$, $g \in C_b^{n-1}(Y)$ be chosen as in the statement of Proposition 4.1. Of course we may extend g to an element $\hat{g} \in C_b^{n-1}(X)$ such that $i^{n-1}(\hat{g}) = g$ and $\|\hat{g}\|_\infty = \|g\|_\infty$ (for example, we may extend g to zero on simplices that are not contained in Y). Let now $f' = f + d^{n-1}\hat{g}$, and let $c \in C_n(X)$ be any representative of α . By point (2) of Proposition 4.1 we have

$$(4.2) \quad f'(c) = (f + d^{n-1}\hat{g})(c) = f(c) + \hat{g}(d_n c) = \|\alpha\|_1(\theta).$$

Point (1) of Proposition 4.1 imply that f' is a relative cocycle. We denote by $[f'] \in H_b^n(X, Y)$ the corresponding relative cohomology class.

Let us now recall that by Theorem 1 the map

$$H(j^n): H_b^n(X, Y) \rightarrow H_b^n(X)$$

is an isometric isomorphism. If $[f]$ denotes the class of f in $H_b^n(X)$, then the equality $f' = f + d^{n-1}\hat{g}$ implies that $H(j^n)([f']) = [f]$. As a consequence of Theorem 1, this gives in turn that

$$\|[f']\|_\infty = \|[f]\|_\infty \leq 1,$$

where the last inequality follows from point (3) of Proposition 4.1. In other words, for every $\varepsilon > 0$ we may find a cochain $f'' \in C_b^{n-1}(X, Y)$ such that $\|f' + d^{n-1}f''\|_\infty \leq 1 + \varepsilon$. Since f'' vanishes on $d_n c \in C_{n-1}(Y)$, using Equation (4.2) we may now conclude that

$$\|\alpha\|_1(\theta) = f'(c) = (f' + d^{n-1}f'')(c) \leq \|f' + d^{n-1}f''\|_\infty \|c\|_1 \leq (1 + \varepsilon)\|c\|_1.$$

Since this inequality holds for every representative $c \in C_n(X)$ of α and for every $\varepsilon > 0$, we finally have that

$$\|\alpha\|_1(\theta) \leq \|\alpha\|_1.$$

This concludes the proof of Theorem 2.

REMARK 4.2. Since the norm $\|\cdot\|_1(\theta)$ on relative chains is equivalent to the usual norm $\|\cdot\|_1 = \|\cdot\|_1(0)$ for every $\theta \in [0, \infty)$, a relative cochain is bounded with respect to $\|\cdot\|_1(\theta)$ if and only if it is bounded with respect to $\|\cdot\|_1$. Therefore, one may endow the set $C_b^\bullet(X, Y)$ with the dual norm $\|\cdot\|_\infty(\theta)$ of $\|\cdot\|_1(\theta)$. This norm induces a seminorm on the relative bounded cohomology module $H^\bullet(X, Y)$, already introduced by Gromov in [5], and still denoted by $\|\cdot\|_\infty(\theta)$. Gromov's original formulation of the Equivalence Theorem contains two statements. The second one (which is the most widely exploited for applications) is just Theorem 2 stated above. The first one asserts that, when all the components of Y have amenable fundamental groups, the seminorm $\|\cdot\|_\infty(\theta)$ on $H_b^n(X, Y)$, $n \geq 2$, does not depend on θ .

5. THURSTON'S SEMINORMS ON RELATIVE HOMOLOGY AND PROOF OF THEOREM 3

Let us describe a family of seminorms on $H_n(X, Y)$ that was introduced by Thurston [12, Section 6.5]. For every $\alpha \in H_n(X, Y)$ and $t > 0$ we set

$$\|\alpha\|_{(t)} = \inf\{\|z\|_1 \mid z \in C_n(X, Y), [z] = \alpha, \|d_n z\|_1 \leq t\}.$$

Note that we understand that $\inf \emptyset = +\infty$. Following Thurston⁴, we set

$$\|\alpha\|_{(0)} = \lim_{t \rightarrow 0} \|\alpha\|_{(t)} = \sup_{t > 0} \|\alpha\|_{(t)}.$$

It readily follows from the definitions that Theorem 3 is equivalent to the statement that $\|\alpha\|_1 = \|\alpha\|_{(0)}$ for every $\alpha \in H_n(X, Y)$, $n \geq 2$, provided that the fundamental group of each component of Y is amenable. Therefore, the equivalence between Theorems 2 and 3 is an immediate consequence of the following

⁴The norm $\|\alpha\|_{(0)}$ is denoted by $\|\alpha\|_0$ in [12]. We introduce the parenthesis in the notation for $\|\alpha\|_{(t)}$ to avoid any ambiguity when $t = 1$.

lemma which is stated in [5, page 56] and proved here below for the sake of completeness.

LEMMA 5.1. *The seminorms $\|\cdot\|_1(\infty)$ and $\|\cdot\|_{(0)}$ on $H_n(X, Y)$ coincide.*

Proof. Take $\alpha \in H_n(X, Y)$. For every $\theta \in [0, \infty)$ and $t > 0$ we have

$$\begin{aligned} \|\alpha\|_1(\theta) &\leq \inf_z \{\|z\|_1(\theta) \mid [z] = \alpha, \|d_n z\|_1 \leq t\} \\ &\leq \inf_z \{\|z\|_1 + \theta t \mid [z] = \alpha, \|d_n z\|_1 \leq t\} \\ &= \|\alpha\|_{(t)} + \theta t. \end{aligned}$$

By passing to the limit on the right side for $t \rightarrow 0$ we get $\|\alpha\|_1(\theta) \leq \|\alpha\|_{(0)}$ for every $\theta \in [0, \infty)$, so $\|\alpha\|_1(\infty) \leq \|\alpha\|_{(0)}$.

Let us now prove the other inequality. Of course we may restrict to the case $\|\alpha\|_1(\infty) < \infty$. Let us fix $\epsilon > 0$. By definition there exists a sequence $\{z_i\}_{i \in \mathbb{N}} \subseteq C_n(X)$ such that $[z_i] = \alpha$ and

$$\|z_i\|_1 + i\|d_n z_i\|_1 \leq \|\alpha\|_1(i) + \epsilon \leq \|\alpha\|_1(\infty) + \epsilon.$$

Since $\|\alpha\|_1(\infty) < \infty$ the sequence $\{\|d_n z_i\|_1\}_i$ converges to 0. As a consequence, for every $\delta > 0$ there exists $i_0 \in \mathbb{N}$ such that $\|d_n z_{i_0}\|_1 \leq \delta$, so that

$$\|\alpha\|_{(\delta)} \leq \|z_{i_0}\|_1 \leq \|z_{i_0}\|_1 + i_0\|d_n z_{i_0}\|_1 \leq \|\alpha\|_1(\infty) + \epsilon.$$

Since this estimate holds for every $\delta > 0$, we may pass to the limit for $\delta \rightarrow 0$ and obtain the inequality $\|\alpha\|_{(0)} \leq \|\alpha\|_1(\infty) + \epsilon$, whence the conclusion since ϵ is arbitrary. \square

6. SIMPLICIAL VOLUME OF GENERALIZED DEHN FILLINGS

Let us begin by recalling the definition of generalized Dehn filling [3]. Let $n \geq 3$ and let M be a compact orientable n -manifold such that $\partial M = N_1 \cup \dots \cup N_m$, where N_i is an $(n-1)$ -torus for every i .

For each $i \in \{1, \dots, m\}$ we put on N_i a flat structure, and we choose a totally geodesic k_i -dimensional torus $T_i \subseteq N_i$, where $1 \leq k_i \leq n-2$. Each N_i is foliated by parallel copies of T_i with leaf space L_i which is homeomorphic to a $(n-1-k_i)$ -dimensional torus. The *generalized Dehn filling* $M(T_1, \dots, T_m)$ is defined as the quotient of M obtained by collapsing N_i on L_i for every $i \in \{1, \dots, m\}$. Observe that unless $k_i = 1$ for every i , $M(T_1, \dots, T_m)$ is not a manifold. However, being a pseudomanifold, $M(T_1, \dots, T_m)$ admits a fundamental class, whence a well-defined simplicial volume.

We propose two proofs of Theorem 4. The first proof follows very closely Fujiwara and Manning's approach, which is in turn inspired by Thurston [12]. In fact, in [4] the authors provided both an explicit homological proof of the Equivalence Theorem in the case of manifolds with π_1 -injective toric boundary and residually finite fundamental group, and an explicit proof of the uniform boundary condition for tori. The second alternative short proof of the theorem

relies more directly on the isometry proved in Theorem 1, thus avoiding the explicit use of the Equivalence Theorem.

First proof of Theorem 4. Let us set $N = M(T_1, \dots, T_m)$, $L = \sqcup_{i=1}^m L_i$ and let $p_n: C_n(M, \partial M) \rightarrow C_n(N, L)$ be the map induced by the projection. By Theorem 3, for every $\epsilon > 0$ there exists a relative fundamental cycle $c \in C_n(M, \partial M)$ such that $\|c\|_1 \leq \|M, \partial M\| + \epsilon$ and $\|d_n c\|_1 \leq \epsilon$. Since each leaf space L_i is homeomorphic to a $(n-1-k_i)$ -dimensional torus, the cycle $p_n(d_n c) \in C_{n-1}(L)$ is a boundary. Moreover, since $\pi_1(L)$ is amenable, the module $C_{n-1}(L)$ satisfies Matsumoto-Morita's *uniform boundary condition* [8], so there exist $K > 0$ (independent of c) and $c' \in C_n(L)$ such that $d_n c' = p_n(d_n c)$ and $\|c'\|_1 \leq K \|p_n(d_n c)\|_1$. It is easy to check that $p_n(c) - c'$ is a fundamental cycle for N and

$$\begin{aligned} \|N\| &\leq \|p_n(c) - c'\|_1 \leq \|p_n(c)\|_1 + \|c'\|_1 \\ &\leq \|c\|_1 + \|c'\|_1 \\ &\leq \|M, \partial M\| + \epsilon + K \|p_n(d_n c)\|_1 \\ &\leq \|M, \partial M\| + \epsilon + K\epsilon. \end{aligned}$$

Since ϵ is arbitrary, this concludes the proof. \square

Second proof of Theorem 4. Let $p: (M, \partial M) \rightarrow (N, L)$ be the projection map and $j: (N, \emptyset) \hookrightarrow (N, L)$ be the inclusion map. By the exact sequence of the pair the inclusion map induces an isomorphism $H(j_n): H_n(N) \rightarrow H_n(N, L)$. Since every component of L has amenable fundamental group, Theorem 1 implies that the inclusion map induces an isometric isomorphism $H(j^n): H_b^n(N, L) \rightarrow H_b^n(N)$ in bounded cohomology. Via the translation principle, this implies in turn that also $H(j_n): H_n(N) \rightarrow H_n(N, L)$ is an isometry.

Denoting by ψ_n the inverse of $H(j_n)$, it is easy to verify that $\psi_n(H(p_n)([M, \partial M]))$ is a fundamental class for N . Since $H(p_n)$ is contracting and ψ_n is an isometry, we now have that

$$\begin{aligned} \|N\|_1 &= \|\psi_n(H(p_n)([M, \partial M]))\|_1 = \|H(p_n)([M, \partial M])\|_1 \\ &\leq \|[M, \partial M]\|_1 = \|M, \partial M\|, \end{aligned}$$

which finishes the proof of the theorem. \square

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